Algebraic Self-Similar Renormalization in Theory of Critical Phenomena.

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Abstract

We consider the method of self-similar renormalization for calculating critical temperatures and critical indices. A new optimized variant of the method for an effective summation of asymptotic series is suggested and illustrated by several different examples. The advantage of the method is in combining simplicity with high accuracy.

I. INTRODUCTION

In the theory of critical phenomena one usually obtains critical temperatures and critical indices as expansions in powers of some parameters which, as a rule, are not small. For instance, in the Wilson $\epsilon-$expansion [1] one has $\epsilon = 1$. In the field-theoretical approach [2] the expansion is in powers of the renormalized coupling constant $g \cong 1.4$. Such expansions, as is well known, are asymptotic and lead to reasonable results only in the low orders. The direct use of higher orders makes the results only worse [2,3]. When a *number of terms* in a divergent series is known one may invoke resummation techniques, such as Pade and Páde-Borel ones. While the knowledge of only a *few first terms* does not permit to use these techniques. Thus, one always confronts the problem of how to improve the results of a divergent series having only a few terms.

In the present paper we suggest a solution of this problem advancing a method which has the following peculiarities: (i) It permits to accomplish an effective summation of a divergent series consisting of just a couple of terms, when no other method is applicable; (ii) It is simple and accurate, providing an accuracy not worse than sophisticated Pade and Páde-Borel techniques involving about ten terms, when they are available; (iii) It is regular, unambiguously prescribing the way of action. The method suggested is a variant of the method of self-similar renormalization [4-6]. The latter is a renormalization-group approach using self-similarity of subsequent perturbative terms. Since renormalization group is nothing but a kind of a dynamical system, the approach can be formulated in the language of dynamical theory with the usage of its powerful techniques as well as of those of control theory [6-8]. Then the number of the approximation order plays the role of discrete time. Motion with respect to the latter corresponds to transfer from one approximation to another. This makes it possible to define a dynamical system whose trajectory is bijective to the sequence of approximations. Such a dynamical system with discrete time has been called the approximation cascade, Convergence of a sequence of approximations is equivalent to stability of a dynamical trajectory. The stability and, respectively, convergence are governed

by control functions. The fixed point of a trajectory defines the sought function.

This method has been successfully applied to the eigenvalue problem in quantum mechanics [9,10]. Here we consider the resummation problem for the asymptotic series in the theory of critical phenomena and advance a novel variant of the method, more appropriate for this problem.

II. METHOD OF SELF-SIMILAR RENORMALIZATION

The complete description of the method with the corresponding mathematical foundation can be found in Refs. [4-8]. In this Section we give only the general scheme of the method, which is necessary for explaining the new variant we suggest.

Suppose we are interested in a function $f(x)$ of the variable $x \in (-\infty, \infty)$. Let this function satisfy a complicated equation that cannot be solved exactly. Assume that by means of perturbation theory we can get a sequence $\{p_k(x)\}$ of perturbative approximations $p_k(x)$,where $k = 0, 1, 2...$, enumerates the approximation order. Usually, perturbation sequences are divergent. To extract a meaningful result from a divergent sequence one has to involve the so-called resummation techniques. In the method of self-similar renormalization a divergent sequence can be made convergent by introducing additional functions governing convergence (see [4-8]). These functions, because of their role are called governing or control functions. Let s be a set of such control functions entering into a sequence $\{F_k(x, s)\}$ obtained by a perturbation algorithm.

In addition to introducing the control functions, the main idea of the method of self-similar renormalization is to treat the passage from one approximation to another as a motion with respect to the approximation number $k = 0, 1, 2,...$ This motion is realized in the functional space of the considered function as follows. Define the initial approximation

$$F_0(x, s) = f \tag{1}$$

as an equation for the expansion function $x = x(f, s)$. Substitute the latter back to F_k, so that

$$y_k(f,s) \equiv F_k(x(f,s),s). \tag{2}$$

The relation inverse to (2) is

$$F_k(x,s) = y_k(F_0(x,s),s). \tag{3}$$

Let $\{y_k\}$ form a group of transformations with respect to $k = 0, 1, 2...$ Then the trajectory $\{y_k(f,s)\}$ of this dynamical system, according to definitions (2) and (3), is bijective, that is, in one-to-one correspondence, to the approximation sequence $\{F_k(x,s)\}$. This dynamical system with discrete time k has been called [7,8] the approximation cascade. The attracting fixed point of the cascade trajectory is, by construction, bijective to the limit of the approximation sequence $\{F_k(x,s)\}$, that is, corresponds to the sought function.

To deal with continuous time is easier than with discrete. Therefore, we embed the approximation cascade $\{y_k\}$ into an approximation flow $\{y(t,...)\}$ with continuous time $t \geq 0$. This implies that the trajectory $\{y(t,f,s)\}$ of the flow passes when $t = k = 0, 1, 2, ...,$ through all the points of the cascade trajectory,

$$y(k,f,s) = y_k(f,s) \qquad (k = 0, 1, 2, ...). \tag{4}$$

The evolution equation for the flow reads

$$\frac{\partial}{\partial t} y(t,f,s) = v(y(t,f,s)), \tag{5}$$

with the right-hand side being the velocity field. The latter, in the language of renormalization-group theory, is often called the Gell-Mann-Low or $\beta-$function.

Integrating the evolution equation (5) from $t = k$ to $t = k^*$, we get the *evolution integral*

$$\int_{y_k}^{y^*_{k+1}} \frac{df}{v(f,s)} = k^* - k, \tag{6}$$

in which $y_k = y(k,f,s)$ and $y^*_{k+1} = y(k^*,f,s)$. Before specifying the numbers k and k^* in the limits of the evolution integral, let us note that the differential form (5) of the evolution equation, or its integral form (6), are equivalent to the functional relation

$$y(t+t',f,s) = y(t,y(t',f,s),s). \tag{7}$$

The latter in physical applications is labeled as the self-similarity relation, which explains the term we use. The self-similarity, in general, can occur with respect to motion over different parameters. In our case, this is the motion over the steps of a calculational procedure, the number of steps playing the role of effective time.

If there exists an attractive fixed point of the approximation-flow trajectory, then it is always possible to find a number k^* in the evolution integral (6) such that the upper limit y_k^* would correspond to an expression

$$F_k^*(x,s) \equiv y(k^*, F_0(x,s), s) \tag{8}$$

representing, with the desired accuracy, the sought function $f(x)$. If y_k^* would be an exact fixed point, then (8) would give an exact answer to the problem. However, a fixed point can be reached only after infinite number of steps $k \to \infty$. For a finite number k, the limit y_k^* may represent a fixed point approximately, because of which it is named the quasi -fixed point. Our aim is to reach the latter as fast as possible, that is, during the minimal time

$$t_k^* = \min(k^* - k), \tag{9}$$

or the minimal number of steps. When there are no additional restrictions, the minimal number of steps counted by k is 1, so that

$$abs \, \min \, t_k^* = 1. \tag{10}$$

In the case when some constraints are imposed on the motion, the minimal time (9) should correspond to the conditional minimum. For instance, if a value $f_0 \equiv f(x_0)$ of the sought function $f(x)$ is given for some $x_{0,}$ then we can find t_k^* by requiring the trajectory of the approximation cascade to pass through the given point $f_{0.}$.

To calculate the evolution integral (6), we need to define the velocity field. This can be done by the Euler discretization of (5) yielding the finite-difference form

$$v_k(f,s) = y_k(f,s) - y_{k-1}(f,s). \tag{11}$$

Substituting (11) into (6), and using (3), we come to the representation

$$\int_{F_k}^{F^*_{k+1}} \frac{df}{v_{k+1}(f,s)} = t^*_k, \tag{12}$$

for the evolution integral (6), where $F_k = F_k(x,s),\ F^*_{k+1} = F^*_{k+1}(x,s)$.

Finally, we have to define the set s of control functions. The role of the latter is to govern the convergence of the approximation sequence. This convergence can be expressed, in the language of dynamical theory, as the stability of the cascade trajectory. A useful tool for analyzing stability is a set $\{\mu_k\}$ of the local multipliers

$$\mu_k(f,s) = \frac{\partial}{\partial f} y_k(f,s). \tag{13}$$

The inequality

$$\mid \mu_k(f,s) \mid < 1 \tag{14}$$

is the condition of local stability at the step k with respect to the variation of an initial point f. The equality $\mid \mu_k(f,s) \mid = 1$ implies local neutral stability. For a convergent sequence corresponding to a contracting mapping, the condition of asymptotic stability is

$$\mid \mu_k(f,s) \mid \to 0 \qquad (k \to \infty). \tag{15}$$

The approximation cascade $\{y_k\}$ describes the motion in the functional space $\{f\}$. To return to the domain $\{x\}$, we must use the inverse transformation (3). With the help of the latter we may pass from the multiplier (13) given on the functional space $\{f\}$ to its image

$$m_k(x,s) = \mu_k(F_0(x,s),s) \tag{16}$$

being a function of x. For the image (16), the same stability condition as in (14) can be written,

$$\mid m_k(x,s) \mid < 1. \tag{17}$$

According to (15), the local multipliers diminish when approaching an attracting fixed point. That is, the variation of initial condition f produces lesser and lesser effect on the trajectory

as soon as the attractor becomes closer. In other words, the lesser are the absolute values of multipliers, the more stable is the trajectory. Therefore, it is reasonable to define the control functions as those minimizing the absolute values of the local multipliers, making by this the trajectory more stable at each step k. In this way, a set s of control functions is to be defined by the *principle of maximal stability* written as

$$\mid m_k(x, s_k(x)) \mid = \min_s \mid m_k(x, s) \mid . \tag{18}$$

Because of this, the control functions $s_k(x)$ defined by the principle (18) may be called the *stabilizing functions* or *stabilizers.*

Note that the control functions may be introduced in several ways, as is discussed in Refs.[4-10], however always being related to stability conditions and the closeness of a trajectory to an attracting fixed point. In all the cases the control functions are to be defined so that they could accomplish their main job, i.e., to govern the convergence of an approximation sequence, which, in the terms of dynamical theory is equivalent to stabilizing the cascade trajectory. In the present paper we shall use the definition of stabilizers given in (18).

After the stabilizers are defined, we have to substitute them into the corresponding approximations $F_k(x, s)$ getting

$$f_k(x) \equiv F_k(x, s_k(x)). \tag{19}$$

This stage can be called the *stabilizing renormalization* of a perturbative sequence.

Then, considering the motion near the renormalized quantity (19) by means of the evolution integral (12), we obtain

$$f_k^*(x) \equiv F_k^*(x, s_k(x)). \tag{20}$$

This step can be called the *dynamical renormalization***.** And the whole procedure of the double renormalization (19) and (20) is named the self-similar renormalization. It is worth noting that the evolution equation (5) is, generally, nonlinear and can have several different

solutions leading to different self-similar approximations (20). In such a case, to select a physically meaningful solution, we need to involve additional conditions as constraints.The role of the latter can be played, e.g., by properties of symmetry, asymptotic properties at $x \rightarrow 0$ or $x \rightarrow \infty$, sum rules or other relations containing some known information on the character of the sought solution. Such additional constraints narrow the set of possible solutions to a class with desired properties. Thus, we should always remember from what class we are looking for a solution.

Keeping in mind that we wish to get a good accuracy for the sought function, having just a few perturbative terms available, we need to find out some tricks which could effectively increase perturbation order. We suggest below one such trick.

Suppose that there is a sequence of approximations $p_k(x)$ having polynomial structure, k showing the order of the polynomial. This order can be effectively increased by means of the multiplicative transformation

$$P_k(x,s) = x^s \; p_k(x), \;\; s \geq 0. \tag{21}$$

Then, the order of the expression (21) becomes $k + s$. The transformation inverse to (21), as is evident, is

$$p_k(x) = x^{-s} \; P_k(x,s). \tag{22}$$

Following the method described above, we consider the sequence $\{P_k(x,s)\}$ and construct an approximation cascade $\{y_k\}$ whose trajectory $\{y_k(f,s)\}$ is bijective to $\{P_k(x,s)\}$. Solving the evolution integral (12), we have $P_k^*(x,s)$. From the principle of maximal stability (18) we define the stabilizers $s_k(x)$. Substituting these into $P_k^*(x,s)$ and invoking the inverse transformation (22), we obtain the self-similar approximation

$$f_k^*(x) = x^{-s_k(x)} \; P_k^*(x, s_k(x)). \tag{23}$$

The multiplicative transformation (21) looks as the most natural for the case when the perturbative approximations $p_k(x)$ have the form of polynomials or series, generally speaking,

not necessarily in integer powers. The factor x^s effectively increases the approximation order, and s plays simultaneously the role of stabilizer.

What power s we have to choose, that is, to what effective order we need to go is dictated by the principle of maximal stability selecting the most stable trajectory of the approximations cascade. In particular, it may happen that $s = 0$, and we do not need to proceed further, or, vice versa, we may have to go to the limit of $s \rightarrow \infty$, thus making allowance for all approximation orders. In each concrete case, an effective order which we need to reach depends on how good is the perturbative sequence $\{p_k(x)\}$ we start with and, respectively, how much information can be extracted from its first terms by means of the double renormalization (19) and (20).

The optimization by introducing the stabilizing control functions into the powers of perturbative polynomials renormalizes the algebraic structure of the latter. Because of this, and in order to distinguish the suggested optimization procedure from other possible variants, we shall call it the *algebraic self-similar renormalization*.

To concretize the procedure, let us write explicitly

$$p_k(x) = \sum_{n=0}^{k} a_n \; x^n, \quad a_n \neq 0, \tag{24}$$

as a polynomial of the order k. Following (21) define

$$P_k(x, s) = \sum_{n=0}^{k} a_n \; x^{n+s}. \tag{25}$$

Similarly to (1), we have

$$P_0(x, s) = a_0 \; x^s = f, \tag{26}$$

from where the expansion function is

$$x(f, s) = (\frac{f}{a_0})^{1/s}. \tag{27}$$

The definition (2) yields the points

$$y_k(f, s) = \sum_{n=0}^{k} a_n \; (\frac{f}{a_0})^{n/s+1} \tag{28}$$

of the approximation-cascade trajectory. For the velocity field (11) we get

$$v_{k+1}(f,s) = a_{k+1}(\frac{f}{a_0})^{\frac{k+1}{s}+1}. \tag{29}$$

From the evolution integral (12) we find

$$P^*_{k+1} = \frac{P_k}{(1 - \frac{(k+1)\ a_{k+1}\ t^*_k}{s\ a_0^{\frac{k+1}{s}+1}} P_k^{\frac{k+1}{s}})^{\frac{s}{k+1}}}. \tag{30}$$

The multiplier (13) becomes

$$\mu_k(f,s) = \sum_{n=0}^{k} \frac{a_n}{a_0}\ (1 + \frac{n}{s})\ (\frac{f}{a_0})^{\frac{n}{s}}, \tag{31}$$

and its image (16) reads

$$m_k(x,s) = \sum_{n=0}^{k} \frac{a_n}{a_0}\ (1 + \frac{n}{s})\ x^n. \tag{32}$$

The principle of maximal stability (18) defines the stabilizers $s_k(x)$, whose explicit expressions depend on the coefficients a_n. According to the transformations (21)-(23), from (30) we obtain

$$f^*_{k+1} = \frac{p_k(x)}{(1 - \frac{(k+1)\ a_{k+1}\ t^*_k}{s\ a_0^{\frac{k+1}{s}+1}} x^{k+1} p_k(x)^{\frac{k+1}{s}})^{\frac{s}{k+1}}}, \tag{33}$$

where $s_k(x)$ defines the most stable trajectory. When there are no additional conditions, the minimal value $t^*_k = 1$, as in (10).

As is noted above, it may happen that the most stable trajectory corresponds to $s \to \infty$. Let us show how the self-similar approximation (33) simplifies in this case. It is straightforward to check that the limit of the right side in (33), as $s \to \infty$, leads to

$$f^*_{k+1}(x) = p_k(x) \exp(\frac{a_{k+1}}{a_0} x^{k+1}). \tag{34}$$

One may notice that renormalizing $p_k(x)$ in (34) we can obtain the recurrence relation

$$f^*_{k+1}(x) = f^*_k(x) \exp(\frac{a_{k+1}}{a_0} x^{k+1}). \tag{35}$$

It is possible also to derive several other relations permitting to repeat the self-similar renormalization several times, which is useful when working with high-order terms. However, in what follows we shall limit the consideration of particular examples by keeping only a few terms of the corresponding perturbative series. This is to emphasize that the method suggested allows to reach good accuracy with a minimal number of perturbative terms, when no other resummation technique is applicable. Comparing (33) with (34), we see that the self-similar renormalization can yield quite different expressions, from the fractional form to exponential one. Below we shall illustrate this by some simple examples. Each appearing form of an approximation results from choosing the most stable trajectory by which it is possible to reach a quasi-fixed point during the minimal time. Recall in this connection the analogy with classical mechanics. Notice also that it is possible to follow a trajectory which is stabilized by imposing additional conditions, such as asymptotic properties, or prescribing that the trajectory is to pass through some given points. In such a case the motion will not, generally, be accomplished during the absolute minimal time (10), but the latter should be defined from the additional constraints imposed. All these variants will be exemplified in the following sections.

III. ILLUSTRATION BY SIMPLE EXAMPLES

Suppose that by perturbation theory we have got

$$p_1(x) = 1 - x, \tag{36}$$

with $0 \leq x \ll 1$. How could one continue this expression from small $x \ll 1$ to $x \geq 1$, when no other information is available?

Following the algebraic self-similar renormalization, we construct the transformed polynomial (21) or (25) , which for the case (36) is

$$P_1(x, s) = x^s - x^{1+s}. \tag{37}$$

According to (26), we have the expansion function

$$x(f,s) = f^{1/s}. \tag{38}$$

Then, Eq.(28) gives

$$y_1(f,s) = f - f^{1+\frac{1}{s}}. \tag{39}$$

The velocity field (29) becomes

$$v_1(f,s) = -f^{1+\frac{1}{s}}. \tag{40}$$

The evolution integral (12), leading to (30) now yields

$$P_1^*(x,s) = (\frac{sx}{s+x})^s. \tag{41}$$

For the multiplier (32) we have

$$m_1(x,s) = 1 - (1 + \frac{1}{s})\, x. \tag{42}$$

Minimizing the absolute value of the latter gives the stabilizer

$$\begin{aligned} s_1(x) = \tfrac{x}{1-x}, \qquad & 0 \le x \le 1, \\ s_1(x) \to \infty, \qquad & x \ge 1. \end{aligned} \tag{43}$$

The self-similar approximation (33) reduces to

$$f_1^*(x) = (\frac{s_1(x)}{s_1(x)+x})^{s_1(x)}. \tag{44}$$

Being interested in the region $x \geq 1$ we have to take the limit $s_1(x) \to \infty$. Therefore, for the self-similar approximation (44) we obtain

$$f_1^*(x) = \lim_{s\to\infty} (\frac{s}{s+x})^s = e^{-x}. \tag{45}$$

In the same way, a linear expansion

$$p_1(x) = a_0 + a_1 x, \tag{46}$$

with $a_0,\, a_1 \neq 0$, derived for $\mid x \mid \ll 1$, can be continued to the region $\mid x \mid \geq \mid \frac{a_0}{a_1} \mid$ where it is represented by the self-similar approximation

$$f_1^*(x) = a_0 \exp(\frac{a_1}{a_0} x). \tag{47}$$

Thus, we may conclude that the exponential (47) is a general self-similar representation of a linear expansion (46), when *no additional constraints are imposed.*

Now turn to the case, when we want to construct a self-similar continuation of (36) satisfying the prescribed asymptotic behavior

$$f(x) \propto x^n, \qquad x \to \infty, \tag{48}$$

where $n > 0$ is fixed. Repeating the same procedure as earlier, we come to (44). Comparing the latter with (48), we get $s_1(x) = n$, so that

$$f_1^*(x) = (\frac{n}{n+x})^n. \tag{49}$$

Generalizing this result for a linear combination (46) under the asymptotic condition (48), we have

$$f_1^*(x) = a_0 (\frac{na_0}{na_0 - a_1 x})^n. \tag{50}$$

In this way, one perturbative expansion may have several self-similar representations corresponding to different imposed constraints. The form of these representations can vary between the exponential, (47), and fractional one, (50). However, for each given constraint this form is uniquely defined. If no constraints are imposed, the form of the resulting self-similar approximation is governed by the stabilizers obtained from the principle of maximal stability of a self-similar trajectory.

Let us illustrate how accurate is a self-similar approximation and how it is possible to increase the accuracy by considering higher-order terms of a perturbative expansion. For this purpose take the function $\ln(1 + x)$ with $x \geq 0$. We opt for this function, as an example, since the logarithmic expressions are typical of thermodynamic potentials in statistical mechanics and of generating functionals in field theory.

Write down the three first perturbative approximations for $\ln(1 + x)$ in powers of x, thinking that x is small:

$$
\begin{aligned}
p_1(x) &= x, \\
p_2(x) &= x - \tfrac{x^2}{2}, \\
p_3(x) &= x - \tfrac{x^2}{2} + \tfrac{x^3}{3}.
\end{aligned}
\tag{51}
$$

Our aim is to construct self-similar approximations for $\ln(1+x)$ in the region $x \approx 1$, with expansions (51) that are valid only for $x \ll 1$.

Following the standard prescription of the method, define the transformed polynomials (25) for those in (51), which gives

$$
\begin{aligned}
P_1(x,s) &= x^{1+s}, \\
P_2(x,s) &= x^{1+s} - \tfrac{x^{2+s}}{2}, \\
P_3(x,s) &= x^{1+s} - \tfrac{x^{2+s}}{2} + \tfrac{x^{3+s}}{3}.
\end{aligned}
\tag{52}
$$

Now, the initial, i.e, the lowest order approximation is $P_1(x,s)$. Therefore, as in (26), from the equation $P_1(x,s) = f$ we find the expansion function

$$
x(f,s) = f^{\frac{1}{1+s}}. \tag{53}
$$

The points of the approximation -cascade trajectory (28) are

$$
\begin{aligned}
y_1(f,s) &= f, \\
y_2(f,s) &= y_1(f,s) - \tfrac{1}{2}\, f^{\frac{2+s}{1+s}}, \\
y_3(f,s) &= y_2(f,s) + \tfrac{1}{3}\, f^{\frac{3+s}{1+s}}.
\end{aligned}
\tag{54}
$$

For the velocity field (21) we get

$$
\begin{aligned}
v_2(f,s) &= -\tfrac{1}{2}\, f^{\frac{2+s}{1+s}}, \\
v_3(f,s) &= \tfrac{1}{3}\, f^{\frac{3+s}{1+s}}.
\end{aligned}
\tag{55}
$$

The evolution-integral solutions in (30) become

$$
P_2^*(x,s) = \left[\frac{2\,(1+s)\,x}{2\,(1+s)+x}\right]^{1+s}, \quad P_3^*(x,s) = \left[\frac{1}{x^2\,(1-\frac{x}{2})^{\frac{2}{1+s}}} - \frac{2}{3(1+s)}\right]^{\frac{1+s}{2}}. \tag{56}
$$

And for the multipliers in (32) we have

$$m_2(x,s) = 1 - \tfrac{1}{2}(\tfrac{2+s}{1+s})\ x,$$
$$m_3(x,s) = m_2(x,s) + \tfrac{1}{3}(\tfrac{3+s}{1+s})\ x^2. \tag{57}$$

The stabilizers are to be defined at each step by minimizing the absolute values of the corresponding multipliers in (57). For instance

$$s_2(x) = 0, \qquad 0 \le x \le 1,$$
$$s_2(x) = 2\tfrac{x-1}{2-x}, \qquad 1 \le x \le 2,$$
$$s_2(x) = \infty, \qquad x \ge 2.$$

The corresponding expression for $s_3(x)$ is also easy to find. For the region of interest, where $x \approx 1$, we have

$$s_2(x) = 0, \qquad s_3(x) \to \infty \qquad (x = 1).$$

This leads to the self-similar approximations

$$f_2^*(x) = \tfrac{2x}{1+x},$$
$$f_3^*(x) = x(1 - \tfrac{x}{2})\exp(\tfrac{x^2}{2}), \tag{58}$$

obtained from (56) as in (23).

In order to check the accuracy of (58) as compared to the perturbative expansions (51), define the percentage errors

$$\epsilon_k(x) \equiv \frac{p_k(x) - f(x)}{\mid f(x) \mid} \times 100\% \tag{59}$$

and , respectively,

$$\epsilon_k^*(x) \equiv \frac{f_k^*(x) - f(x)}{\mid f(x) \mid} \times 100\%, \tag{60}$$

where $f(x) = \ln(1+x)$. At the point $x = 1$, the errors (59) and (60), calculated with respect to $f(1) = \ln(2) = 0.693$, are

$$\epsilon_2(1) = -28\%, \quad \epsilon_3(1) = 20\%,$$
$$\epsilon_2^*(1) = -3.8\%, \quad \epsilon_3^*(1) = 0.67\%.$$

As is seen, the accuracy of the self-similar approximations in (58) is an order higher than that of the perturbative expansions in (51), and this accuracy can be increased by taking into account additional perturbative terms.

In this section we considered simple examples in order to make transparent all steps of our method. This will permit us in the following sections to avoid the repetition of the technical details when applying the method to more complicated physical problems.

IV. CALCULATION OF CRITICAL TEMPERATURE FOR 2D AND 3D ISING MODEL

In this Section we calculate the critical temperature T_c of the two-dimensional ($2d$) and three-dimensional ($3d$) Ising model starting from approximate expressions for T_c obtained by the variational-cumulant expansion (VCE)[11]. The convergence of VCE approximations is very slow and even using the seven consecutive approximations one obtain T_c for $2d$ Ising model with the percentage error of about 11%. We use below the simple variant of the self-similar renormalization when the exact value of the sought function is known for some point, namely the knowledge of T_c for the $2d$ Ising model will be used as an optimization condition for the trajectory, determining the optimal number of steps t^*, then used for calculating T_c for the $3d$ Ising model. Although the expressions below are a little complicated for getting the result in an explicit form, we can realize here the numerical variant of the self-similar renormalization , when sought function is obtained implicitly.

We rewrite the expressions for the critical temperature from [11] in terms of a new variable $x = \frac{1}{d}$, where d is the space dimension. Then we calculate $\widetilde{T_c}(x)$, related to T_c as follows: $T_c = \frac{1}{x}\ \widetilde{T_c}(x)$.

Write down the three approximations to $\widetilde{T_c}(x)$,

$$\begin{aligned} \widetilde{T_{c1}}(x) &= 2 - x, \\ \widetilde{T_{c2}}(x) &= \tfrac{12-12x+2}{6-3x}, \\ \widetilde{T_{c3}}(x) &= \tfrac{24-36x+8x^2+5x^3}{12-12x+2x^2}, \end{aligned} \tag{61}$$

which correspond to the second, third and fourth approximations of [11], respectively.

Following the standard approach described above in Section 3, from the equation $\widetilde{T_c}(x) = f$ we find the expansion function $x(f) = 2 - f$. The points of the approximation cascade trajectory are

$$\begin{aligned} y_1(f) &= f, \\ y_2(f) &= \tfrac{2}{3}(\tfrac{f^2+2f-2}{f}), \\ y_3(f) &= -\tfrac{1}{2}(\tfrac{5f^3-38f^2+56f-24}{f^2+2f-2}). \end{aligned} \tag{62}$$

For the velocity field we get

$$v_3(f) = -\frac{1}{6}\frac{(19f^2 - 22f + 4)}{(f^2 + 2f - 2)}\frac{(f-2)^2}{f}.$$

The evolution integral cannot be calculated explicitly, so we expressed $\widetilde{T}^*_{c3}$ implicitly as a function of x and t^* given by the equation

$$\int_{\widetilde{T}_{c2}}^{\widetilde{T_{c3}}^*} \frac{df}{v_3(f)} = t^*,$$

and obtained $\widetilde{T}^*_{c3}$numerically for two different values $t^* = 1$, or $t^* = 1.5515$, corresponding to a non-optimized and optimized variant respectively. The latter, optimizing number of steps, was obtained from the condition $\widetilde{T}^*_{c3}(\frac{1}{2}, t^*) = \frac{T_{2d}}{2}$, where $T_{2d} = 2.269$ is the celebrated Onsager temperature for the $2d$ Ising model.

At $t^* = 1$ and $d = 2$ we found that $T_c^* = 2.531$ and the percentage error of our estimate is $\epsilon_3^*(T_c) = 11.542\%$, approaching the percentage error reached using seven consecutive approximations to T_c [11].

We should point out here that the authors of [11] did not attempt to calculate T_c for $3d$ Ising model, where the best known "exact" numerical value of the critical temperature is $T_c = 4.51$ [12]. We obtained $T_c^* = 4.712$ at $t^* = 1$ and $d = 3$. The percentage error is $\epsilon_3^*(T_c) = 4.47\%$. For the optimized $t^* = 1.5515$ our estimate of $T_c^* = 4.548$ with the error equal to 0.838% is quite accurate.

It is worth noting that, in principle, the parameter $x = \frac{1}{d}$ may be considered as "small" for $d = 3$ and the expressions for $\widetilde{T_c}(x)$can be expanded in powers of x, thus presenting T_c

in the form of "$\frac{1}{d}-$expansion". We performed the same renormalization procedure as above for

$$\widetilde{T_{c2}}(x) \approx 2 - x - \frac{x^2}{6},$$

$$\widetilde{T_{c3}}(x) \approx 2 - x - \frac{2x^2}{3} - \frac{x^3}{12},$$

and found that in $2d$ for $t^* = 1$, $T_c^* = 2.453$ with $\epsilon_3^*(T_c) = 8.109\%$ and in $3d$-case with $t^* = 1$, we have $T_c^* = 4.701$ with $\epsilon_3^*(T_c) = 8.109\%$.

For the optimized $t^* = 1.241$ and $d = 3$, $T_c^* = 4.624$ and $\epsilon_3^*(T_c) = 2.527\%$. We see that the renormalized $\frac{1}{d}-$expansion gives by order of magnitude the same accuracy that is reached from the renormalized original expressions.

One can conclude from the results presented in this Section, that both the rate of convergence and accuracy of the VCE are greatly improved by applying the self-similar renormalization to the starting VCE approximations. Situation encountered while renormalizing T_c was somewhat very simple, since we possessed three reasonable approximations for renormalization and also knew the exact value of T_c for the $2d$ Ising model. In the next Section we meet the case when the number of the terms available are not sufficient for any variant of renormalization discussed above and no exact value for the quantity under consideration is known.

V. RENORMALIZATION OF SHIFTED-POWER EXPANSIONS

In this Section we apply a modified variant of the self-similar renormalization to the "shifted-power expansions" for the critical indices [13]. Shifted-power expansion or $\Delta-$expansion is of particular interest for systems without upper critical dimensionality. It can be applied to the calculation of critical exponents of a system described by a Landau-Ginzburg (LG) Hamiltonian. Within the framework of one of many possible realizations of the method, the leading nonlinear term of the LG Hamiltonian,ψ^4, is replaced by $(\psi^2)^{3-\Delta}$, then Δ is used as an expansion parameter and at the end of calculations one should set $\Delta = 1$. The expressions for critical indices η and ν were obtained in the following form:

$$\eta = b_2(n)\ \Delta^2, \qquad b_2(n) = \tfrac{(n+4)(n+2)}{48(3n+22)^2},$$
$$\nu = a_0 + a_2(n)\ \Delta^2, \qquad a_0 = \tfrac{1}{2},\ a_2(n) = \tfrac{(n+4)(n+2)}{12(3n+22)^2}, \tag{63}$$

where n is the number of the components of the order parameter.

The corrections to the mean-field values given by (63) are about two orders of magnitude too small. The authors of [13] had noticed that the results are strongly influenced by the ψ^6 interaction and remain too distant from the analyzed ψ^4 behavior. This conclusion agrees well with the rigorous results of [14] where it was shown that special Gaussian points $n = -4, -2$ should appear when the ψ^6 model is considered, while for the ψ^4 model only the point $n = -2$ exists. Appearance in the expressions (63) of the combination $n+4$ signalizes that the critical indices in the interesting physical region $n = 0, 1...3$ are influenced by the point that does not have any meaning for the ψ^4 model at all. Nevertheless, the Gaussian point $n = -2$ does have physical meaning for the ψ^4 model [15-17] and, formally, the results given by (63) are correct at $n = -2$. Therefore, one may hope that a systematic and pernicious influence of the point $n = -4$ can be weakened by some renormalization procedure, at least when a physical region not very distant from the point $n = -2$ is considered. We should stress also that the expressions in (63) do not obey another rigorously studied limit of the LG Hamiltonian, when $n \to \infty$, i.e. $\eta(n \to \infty) \neq 0$ and $\nu(n \to \infty) \neq 1$ [18]. Therefore we do not expect that a successful renormalization of (63) could be realized for n very distant from the point $n = -2$.

Our approach to the renormalization of η and ν should vary, since for the index ν the two terms in the $\Delta-$expansion are available and the renormalization could be carried on straightforwardly, while for the index η only single term was obtained and it is not possible to proceed with extra assumptions.

Write down the two consecutive approximations to the index ν in powers of Δ :

$$\nu_0(n) = a_0,$$
$$\nu_2(n) = a_0 + a_2(n)\ \Delta^2,$$

and apply the same procedure that leads to the expression (47) with a substitution of x to

Δ^2. We also retain in the final expression for the renormalized critical index ν_2^* an effective time t^* which will be exploited as an optimization parameter:

$$\nu_2^*(\Delta, n, t^*) = a_0 \exp\{\frac{a_2(n)}{a_0}\Delta^2 t^*\}. \tag{64}$$

Setting here $\Delta = 1$ and $t^* = 1$ we see that no considerable effect was achieved and the index ν remains in the interval $\nu = 0.5$ to 0.509 while n varies from $n = -2$ to infinity. Impose now an additional condition that in the case of $n = 0$, corresponding to the random walk problem [19], the approximation cascade trajectory pass through the value of the critical index $\nu = 0.588$ known approximately, but with very high accuracy for this physical problem [20]. The choice of this point for optimization of the trajectory is dictated also by the desire to receive renormalized values for the physically interesting region $n = 1, 2, 3$ of "true" phase transitions using information only from the physically distant region, where the random walk problem may be a good choice because it does not correspond to a "true" phase transition, but only a formal analogy exists with the $n \to 0$ limit of the LG Hamiltonian.

So, from the condition $\nu_2^*(1, 0, t^*) = 0.588$ we obtain $t^* = 59$ and the following values at $n = 1, 2; 3$:

$$\nu_2^*(1, 1, t^*) = 0.633, \quad \nu_2^*(1, 2, t^*) = 0.676, \quad \nu_2^*(1, 3, t^*) = 0.715.$$

These values are reasonable as compared to the experiment, high-temperature series, and Borel-summation results [21]. We should point out that only by means of a single parameter we obtain simultaneously renormalized values for the physically interesting situation, i.e a systematic multiplicative error in the initial expansion, caused by peculiarities of the $\Delta-$expansion can be eliminated by a single renormalization step.

It is also possible to find optimal t^* from the condition restoring the correct value of ν at $n \to \infty$, where, as is shown rigorously, $\nu = 1$. In our case, the results happen to be much better than for the initial $\Delta-$expansion, but still largely underestimate ν, e.g at $n = 0$, $\nu^* = 0.544$. Nevertheless , this variant of optimization is of general interest because often the limit $n \to \infty$ (or $d \to \infty$) is well known.

In the case of the index η one should proceed differently, since only a single term in the $\Delta-$expansion is known: we added to the expression for η the term linear in Δ with some yet unknown positive coefficient b defining thus a new quantity $\widetilde{\eta}$. Then carried out the renormalization procedure for $\widetilde{\eta}$, repeating the steps leading to Eq.35. From the renormalized quantity $\widetilde{\eta}^*$ using the variational condition $\frac{\partial\widetilde{\eta}^*}{\partial b}=0$, we determine b as a function of $n,\ \Delta,\ t^*$. Finally, we subtracted the term $b\Delta$ from $\widetilde{\eta}^*$ to find η^*. Following this prescription define

$$\widetilde{\eta}=b\Delta+b_2(n)\Delta^2,$$

then find

$$\widetilde{\eta}^*=b\Delta\exp\{\frac{b_2(n)}{b}\Delta\ t^*\}.$$

This yields

$$\eta^*=\widetilde{\eta}^*-b\Delta=t^*b_2(n)(e-1)\ \Delta^2.$$

At $t^*=1$ the results still remain too small. Imposing an additional optimizing condition on t^* by analogy with the case of ν, that η^* should equal to 0.026, where the critical index η for the random walk problem is taken from [20], we find that $t^*=44$ and

$$\eta^*(n=1,t^*)=0.038,\quad \eta^*(n=2,t^*)=0.048,\quad \eta^*(n=3,t^*)=0.057.$$

These results overestimate η, especially for $n=3$, but are much more realistic than the initial value $\eta\sim 10^{-4}-10^{-3}$. This systematic error can be understood if to notice that already the initial $\Delta-$expansion does not obey the limit of $n\to\infty$, and $\eta(\infty)=0.002$ instead of zero. This systematic deviation cannot be fully corrected by a variational-renormalization procedure. The same is true in the case of index ν, but in this case more information is available from the initial $\Delta-$expansion and the results of renormalization remain reasonable even at $n=3$.

We conclude that by applying the self-similar renormalization to the $\Delta-$expansion for the critical indices, one can obtain reasonable estimates for η and ν, although further improvement of these estimates does not seem plausible, since initial expressions violate an exact relation in the $n\to\infty$ limit and possess an unphysical Gaussian point at $n=-4$.

In the next Section we meet the case when both limits at $n = -2$ and $n \to \infty$ are violated.

VI. IMPORTANCE OF ASYMPTOTIC PROPERTIES

We have seen in the previous Section that the $\Delta-$expansion mimicking the widely accepted Wilson $\epsilon-$expansion is, in the best case, a crude approximation for the critical indices, since an important $n \to \infty$ limit is violated already in the starting terms of the $\Delta-$expansion. The question naturally arises whether the $\epsilon-$expansion obeys exact limits for critical indices, namely at $n = -2$ and $n \to \infty$? The discussion of this question for all critical indices will be presented later. In this Section, we consider the Wilson $\epsilon-$expansion for the critical index δ, discuss its $n = -2$ and $n \to \infty$ limits, observe that they are violated and suggest the self-similar renormalization approach allowing to restore the correct limiting values for the index δ.

Consider the well-known Wilson $\epsilon-$expansion for the critical index δ [1,18] up to the quadratic terms in ϵ :

$$\delta = 3 + \epsilon + c_2(n)\epsilon^2, \qquad c_2(n) = \frac{1}{2}\frac{n^2 + 14n + 60}{(n+8)^2}. \tag{65}$$

At $n = -2$ and $n \to \infty$, $\delta = 4.5$. From the exact results for $n = -2$ vector model [16] and from the scaling law for the $d = 3$ the result $\delta = 5$ follows. The same value $\delta = 5$ was obtained in the case of the spherical model ($n \to \infty$, $d = 3$) [18]. The percentage error for the critical index δ in these limits is therefore -10%. In the physical region $n = 0, 1...3$ the $\epsilon-$expansion gives $\delta = 4.47 - 4.46$. Unfortunately, there is no much experimental data or theoretical estimates available for the index δ, but if we accept the scaling laws as correct and estimate from these values of δ another critical index $\eta = \frac{5-\delta}{1+\delta}$, which is much better known, both experimentally and theoretically, then we appear in the physical region with $\eta \approx 0.1$, that largely (by three times) overestimates η. One may think that the index δ is underestimated by the Wilson $\epsilon-$expansion. So, we need by means of the self-similar

renormalization procedure to continue the asymptotic expression (65) to the region of $\epsilon \sim 1$ with simultaneous restoration of the incorrect limiting values at $n = -2$ and $n \to \infty$. Introduce the following consecutive approximations to the quantity $\widetilde{\delta} = \delta - 3$:

$$\begin{aligned} \widetilde{\delta_1}(\epsilon) &= \epsilon, \\ \widetilde{\delta_2}(\epsilon) &= \epsilon + c_2(n)\epsilon^2. \end{aligned} \tag{66}$$

By repeating the same steps that led us to (56), we obtain

$$\widetilde{\delta_2}^*(\epsilon, s) = [\frac{\epsilon}{1 - \frac{c_2\epsilon}{1+s}}]^{1+s},$$

and

$$\delta^* = 3 + [\frac{\epsilon}{1 - \frac{c_2\epsilon}{1+s}}]^{1+s}. \tag{67}$$

Setting in (67) $\epsilon = 1$ and $n = -2$, or $n \to \infty$, it is easy to show that only for $s = 0$ both limits can be satisfied! Therefore

$$\delta^* = 3 + \frac{\epsilon}{1 - c_2(n)\epsilon}. \tag{68}$$

The expression (68) in the physical region gives the following values:

$$\begin{aligned} \delta^*(n = 0) = 4.882, \quad \delta^*(n = 1) = 4.862, \\ \delta^*(n = 2) = 4.852, \quad \delta^*(n = 3) = 4.847. \end{aligned}$$

The index η, corresponding to these values can be easily obtained from the scaling law:

$$\begin{aligned} \eta(n = 0) = 0.02, \quad \eta(n = 1) = 0.024, \\ \eta(n = 2) = 0.025, \quad \eta(n = 3) = 0.026. \end{aligned}$$

These results better agree with the general understanding of η as of a small index and are much closer to the results of the Borel summation than the initial $\eta = 0.1$, obtained from the $\epsilon-$expansion (65) and the scaling law.

We conclude that the application of the self-similar renormalization improves the Wilson $\epsilon-$expansion for the index δ both qualitatively and quantitatively. This example stresses once again an importance of obeying different reasonable limits in the expressions for the

critical indices. Another illustration is given in the next Section where the self-similar renormalization of $\frac{1}{n}-$expansion and of the $\epsilon-$expansion around lower critical dimension two (in powers of $d-2$) is considered.

VII. INVERSE LARGE COMPONENT EXPANSION (1/N) AND EXPANSION IN POWERS OF D-2 (2+ϵ).

The large $n-$expansions ($\frac{1}{n}-$expansion) [18,22] and $\epsilon-$expansion around the lower critical dimension two ($d-2-$expansion) [23] had raised high expectations as an alternative to the Wilson $\epsilon-$expansion and field-theoretical approach [2]. Nevertheless they had never became competitive, remaining a useful guide to the region of large n and a good qualitative tool, when different aspects of the two-dimensional behavior are considered. It is clear that the values of critical indices given by $\frac{1}{n}$ and $d-2-$expansion do not obey the $n=-2$ Gaussian limit, becoming divergent at $n=0$ and $n=2$ ($d=3$), respectively. Therefore, it is not possible to get a reasonable estimate for ν and η in the physical region $n=1,2,3$ lying too close to the spurious pole and too far from the correct $n\to\infty$ limit, supported by both expansions.

Consider the $\frac{1}{n}-$expansion for the critical index γ :

$$\gamma = 2 - \frac{24}{\pi^2 n},$$

from where

$$\gamma(n=1) = -0.432, \quad \gamma(n=2) = 0.784, \quad \gamma(n=3) = 1.189.$$

Correspondingly, the two approximations in powers of $\frac{1}{n}$ can be written down

$$\gamma_0(n) = 2,$$
$$\gamma_1(n) = 2 - \tfrac{24}{\pi^2 n}.$$

Proceeding in accordance with the self-similar renormalization prescriptions, the multiplier $m_1(n,s)$ can be found:

$$m_1(n,s) = 1 - \frac{12}{\pi^2}\frac{s-1}{sn}. \tag{69}$$

As is seen, the minimum of $\mid m_1(n,s) \mid$ for $n \geq 2$ is reached for $s \to \infty$. This gives

$$\gamma_1^*(n) = 2\exp(-\frac{12}{\pi^2 n}),$$

and, correspondingly,

$$\gamma_1^*(n=2) = 1.089, \quad \gamma_1^*(n=3) = 1.334.$$

We see that for $n = 3$ the value given by γ_1^* becomes reasonable, deviating from the result of the Borel-summation $\gamma = 1.386$ [20] with the percentage error equal to -3.752%, while the initial $\frac{1}{n}-$expansion has the percentage error of -14.214%. For $n = 1$, the minimum of $\mid m_1(n,s) \mid$ is reached for $s = 5.633$; correspondingly $\gamma_1^*(n=1) = 0.508$. We observe that for $n = 1, 2$ the results are improved if compared to the initial $\frac{1}{n}-$expansion .

The $\frac{1}{n}-$expansion for critical index η is given as follows [18]:

$$\eta = \frac{8}{3\pi^2 n} - (\frac{8}{3})^3 \frac{1}{\pi^4 n^2}. \tag{70}$$

This equation becomes singular at $n = 0$ and negative at $n = -2$, so that, despite its correct by design behavior at $n \to \infty$, the values given by (70) at $n = 1, 2, 3$ are too large:

$$\eta(n=1) = 0.076, \quad \eta(n=2) = 0.086, \quad \eta(n=3) = 0.068.$$

The direct application of the self-similar renormalization using $\frac{1}{n}$ as a renormalization parameter with

$$\eta_1 = \frac{8}{3\pi^2}\frac{1}{n}, \qquad \eta_2 = \frac{8}{3\pi^2}\frac{1}{n} - (\frac{8}{3})^3 \frac{1}{\pi^4}\frac{1}{n^2},$$

as consecutive approximations does not improve the situation, since the influence of singularity at $n = 0$ is too strong. To avoid this divergence, we re-expanded the expression (70) in powers of the parameter $y = \frac{(n+2)}{(n+8)^2}$, expressing n as a function of y :

$$n = \frac{1}{2y}(1 - 16y + \sqrt{1 - 24y}). \tag{71}$$

This choice of the re-expansion parameter is not unique, but the combination $\frac{n+2}{(n+8)^2}$ frequently appears in the Wilson $\epsilon-$expansion. Such a transformation restores the correct value of η at $n=-2$ and also keeps intact the correct limit at $n\to\infty$. Up to the third order in y we obtain:

$$\eta = ay + by^2 + cy^3, \qquad a=\frac{8}{3\pi^2}, \quad b=\frac{112}{3\pi^2}, \quad c=\frac{1856}{3\pi^2}-\frac{14336}{27\pi^4}.$$

Thus the following consecutive approximations may be written down:

$$\begin{aligned}\eta_1(y) &= ay,\\ \eta_2(y) &= ay+by^2, \qquad .\\ \eta_3(y) &= ay+by^2+cy^3.\end{aligned}$$

Proceeding in the usual manner, we obtain

$$\eta_2^*(y) = ay\exp(\frac{b}{a}y), \tag{72}$$

$$\eta_3^*(y) = (ay+by^2)\exp(\frac{c}{a}y^2). \tag{73}$$

Returning to the initial variable we have

$$\begin{aligned}\eta_2^*(n=0)=0.013, \quad \eta_2^*(n=1)=0.016,\\ \eta_2^*(n=2)=0.018, \quad \eta_2^*(n=3)=0.019,\end{aligned}$$

and

$$\begin{aligned}\eta_3^*(n=0)=0.015, \quad \eta_3^*(n=1)=0.02,\\ \eta_3^*(n=2)=0.023, \quad \eta_3^*(n=3)=0.025.\end{aligned}$$

The values of multipliers in these cases are as follows:

$$\begin{aligned}m_2(y,s) &= 1+\tfrac{b}{a}\tfrac{2+s}{1+s}\ y,\\ m_3(y,s) &= m_1(y,s)+\tfrac{c}{a}\tfrac{3+s}{1+s}\ y^2,\end{aligned}$$

These values are very close to each other, e.g for $n=3$, $m_2=1.022$ and $m_3=1.037$. From the stability viewpoint the corresponding approximations are almost equivalent. It is

also possible to improve results for η by applying the second self-similar renormalization, as in the recurrence relation (35). This gives η^* in the form of a continued exponential [24]:

$$\eta^* = ay \exp(\frac{b}{a} y \exp(\frac{c}{b} y)),$$

so that

$$\eta^*(n=0) = 0.016, \quad \eta^*(n=1) = 0.024,$$
$$\eta^*(n=2) = 0.03, \quad \eta^*(n=3) = 0.032.$$

These values, especially for $n = 2, 3$, are quite reasonable. The results for $n = 0$, not surprisingly, remain too small, since we used for the renormalization procedure only the large n expansion, obviously too short of information about the limit for small n. In order to weaken the influence of the particular way of defining coefficients a, b, c, it is possible to proceed with a variational-optimization procedure, considering η^* as a function of two unknown parameters $\overline{a},\ \overline{b}$ and determining them from the conditions

$$\frac{\partial \eta^*}{\partial \overline{a}} = 0, \qquad \frac{\partial \eta^*}{\partial \overline{b}} = 0.$$

For the particular choice

$$\eta^* = \overline{a} y \exp(\frac{\overline{b}}{\overline{a}} y \frac{1}{1 - \frac{c}{\overline{b}} y}),$$

we find that $\eta^* = 4ecy^3$ and

$$\eta^*(n=0) = 0.019, \quad \eta^*(n=1) = 0.032,$$
$$\eta^*(n=2) = 0.04, \quad \eta^*(n=3) = 0.043.$$

The $\epsilon-$expansion with $\epsilon = d-2$ around the lower critical dimension for the critical index ν, is written in the form

$$\nu^{-1} = d - 2 + \frac{(d-2)^2}{n-2} \tag{74}$$

for $d > 2$ and $n > 2$. At $n = 3,\ d = 3,\ \nu = 0.5$, giving rather crude estimate coinciding with the mean-field result. The self-similar renormalization using ϵ as a parameter for renormalization does not improve this value, because $\nu \to \infty$ in the starting point $d = 2,\ n = 3$. On the other hand, at $n \to \infty,\ d = 3$, formula (74) gives $\nu = 1$, i.e the correct

limiting value known rigorously for the spherical model. Re-expanding (74) in powers of $\frac{1}{n}$ around this correct value we obtain

$$\nu^{-1} = 1 + \frac{1}{n} + 2\frac{1}{n^2} + ... \tag{75}$$

Proceeding in accordance with the standard prescription and using $\frac{1}{n}$ as a renormalization parameter, we define

$$\nu_0^{-1} = 1,$$
$$\nu_1^{-1}(n) = 1 + \tfrac{1}{n}.$$

Then, we readily obtain for $\nu_1^*(n)$ the following expression

$$\nu_1^*(n) = \exp(-\frac{1}{n})$$

and $\nu_1^*(n=3) = 0.717$, giving reasonable estimate for the critical index ν. The percentage error, when compared to the result of the Borel summation $\nu = 0.705$ [20] equals 1.702%, while for the initial $d-2-$expansion it equals -29.078%.

For the critical index η, the $d-2-$expansion has the following form [23]:

$$\eta = a(n)(d-2) - b(n)(d-2)^2, \qquad a(n) = \frac{1}{n-2}, \quad b(n) = \frac{n-1}{(n-2)^2}.$$

At $n=3,\ d=3$, this gives $\eta = -1$, in disagreement with all known about this index.

We use below $d-2=\epsilon$ as a renormalization parameter, since at $d=2,\ \eta=0$, being a reasonable starting point for the trajectory. Define the following approximations to η :

$$\eta_1(\epsilon, n) = a(n)\epsilon,$$
$$\eta_2(\epsilon, n) = \eta_1(\epsilon, n) - b(n)\epsilon^2.$$

The multiplier $m_2(\epsilon, n) = 1 - \frac{b(n)}{a(n)}\frac{2+s}{1+s}\epsilon$ reaches its minimum at $s \to \infty$, therefore

$$\eta_2^*(\epsilon, n) = a(n)\epsilon \exp\{-\frac{b(n)}{a(n)}\epsilon\},$$

and $\eta_2^*(1,3) = 0.135$, which is a considerable improvement if compared to $\eta = -1$ from the $d-2$-expansion.

We conclude that the self-similar renormalization improves the quality of estimates also for $\frac{1}{n}$ and $d-2$-expansions, achieving the quantitative agreement with other approaches. However, the broken $n=-2$ Gaussian limit still makes the possibilities of improving the results very narrow, usually an improvement is achieved for $n=3$, but not for lower n.

In the next Section we consider an opposite case of an expansion obeying only $n=-2$ limit, but with broken limit at $n\to\infty$. Such situation is similar to that of Section 5, but no non-physical Gaussian points will be present.

VIII. SELF-AVOIDING WALK PROBLEM (N+2-EXPANSION FOR N=0)

Interesting properties of the LG model for $n=-2$ have been analyzed in a number of works [14-17]. Physically, $n=-2$ corresponds to a Gaussian polymer with the exponents $\gamma=1,\ \eta=0,\ \nu=1/2$. From the scaling laws in $d=3$ one can see that $\alpha=\frac{1}{2},\ \delta=5,\ \beta=\frac{1}{4}$. It seems natural to develop expansions in powers of $n+2$ ($n+2-$expansion) around this well defined limit [16]. To our knowledge, this idea has never been put into practice. The task of obtaining the $n+2-$expansion is simplified if we note that the Wilson $\epsilon-$expansion for the critical indices $\gamma,\eta,\nu,\alpha,\beta$ obeys the $n=-2$ limiting values. In order to obtain the $n+2-$expansion we simply re-expand the Wilson $\epsilon-$expansion at $\epsilon=1$ in powers of $n+2$. Of course, $n+2-$expansion could be derived also from the "first principles" in a way similar to the $\epsilon-$expansion , or $\frac{1}{n}-$expansion. The nearest to the point $n=-2$ physically interesting case is located at $n=0$, corresponding to the self-avoiding walk problem equivalent to a polymer. We believe that the case of the order parameter dimensionality of $n=-2$ and $n=0$ are closely connected in a way similar to the connection existing between the space dimensionalities $d=4$ and $d=3$, with the only difference that the $\epsilon-$expansion is substituted by the $n+2-$expansion. We apply below the self-similar renormalization to the $n+2-$expansion for the critical indices, presenting only the results for $n=0$. The values of the critical indices are not as good for $n=1,2,3$. The expansion parameter is too large in the latter cases and also the $n\to\infty$ limit is violated, so that the trajectory strongly

deviates for larger n from the correct but distant starting point.

Up to the second order in ϵ, the critical index ν is given as follows:

$$\nu = \frac{1}{2} + \frac{n+2}{4(n+8)}\epsilon + \frac{n+2}{8(n+8)^3}(n^2+23n+60)\epsilon^2. \tag{76}$$

Put here $\epsilon = 1$ and expand (76) in powers of $z = n+2$, up to the third order terms in z :

$$\nu = \frac{1}{2} + a_1 z + a_2 z^2 + a_3 z^3; \qquad a_1 = \frac{5}{96}, \quad a_2 = -\frac{1}{864}, \quad a_3 = -\frac{7}{3456}.$$

The following approximations to the quantity $\widetilde{\nu} = \nu - \frac{1}{2}$ can be readily written down:

$$\begin{aligned} \widetilde{\nu_1}(z) &= a_1 z, \\ \widetilde{\nu_2}(z) &= a_1 z + a_2 z^2, \\ \widetilde{\nu_3}(z) &= a_1 z + a_2 z^2 + a_3 z^3. \end{aligned} \tag{77}$$

Following the standard way, we find the multipliers

$$\begin{aligned} m_2(z,s) &= 1 + \tfrac{a_2}{a_1}\tfrac{2+s}{1+s} z, \\ m_3(z,s) &= m_2(z,s) + \tfrac{a_3}{a_2}\tfrac{3+s}{1+s} z^2. \end{aligned} \tag{78}$$

Both multipliers at the point $z = 2$ ($n = 0$), reach their minimum at $s = 0$, where $m_2(2,0) = 0.911, \quad m_3(2,0) = 0.444$. Consequently, the trajectory restored using all three approximations from (77) will be more stable, than that restored from only two approximations, both trajectories being stable. The evolution integral (12) gives

$$\widetilde{\nu_2}^*(z) = a_1 z \frac{1}{1 - \frac{a_2}{a_1} z}, \qquad \widetilde{\nu_3}^*(z) = \frac{\widetilde{\nu_2(z)}}{(1 - 2\frac{a_3}{a_1^3}\widetilde{\nu_2(z)}^2)^{\frac{1}{2}}},$$

and $\nu_2^*(z=2) = 0.6$, $\nu_3^*(z=2) = 0.588$. The former value is exactly the Flory "mean-field" exponent [25], and the latter is the same as the Borel-summation result, considered as the best known estimate for polymers [20]. The latter value $\nu_3^* = 0.588$ should be trusted more, since it is obtained moving along the more stable trajectory than ν_2^*. It is encouraging that the Flory and field-theory results, in our consideration come out as successive approximations.

The Wilson $\epsilon-$expansion for the critical index β, up to the second order in ϵ, is

$$\beta = \frac{1}{2} - \frac{3}{2(n+8)}\epsilon + \frac{(n+2)(2n+1)}{2(n+8)^3}\epsilon^2. \tag{79}$$

So, the $n+2-$expansion, up to the third order, becomes

$$\beta = a_0 + a_1 z + a_2 z^2 + a_3 z^3; \qquad a_0 = \frac{1}{4}, \quad a_1 = \frac{5}{144}, \quad a_2 = \frac{1}{864}, \quad a_3 = -\frac{1}{432}.$$

This results in the following approximations for β :

$$\begin{aligned} \beta_0(z) &= a_0, \\ \beta_1(z) &= \beta_0(z) + a_1 z, \\ \beta_2(z) &= \beta_1(z) + a_2 z^2, \\ \beta_3(z) &= \beta_2(z) + a_3 z^3. \end{aligned} \tag{80}$$

The multipliers $m_1(z,s),\ m_2(z,s),\ m_3(z,s)$ for $z=2$ reach their minima at $s \to \infty$; so that $m_3(2,\infty) < m_2(2,\infty) < m_1(2,\infty)$. The evolution-integral solution for $\beta_3^*(z)$ is

$$\beta_3^*(z) = \beta_2(z)\exp(\frac{a_3}{a_0}z^3), \tag{81}$$

and $\beta_3^*(z=2) = 0.301$. This coincides with the result of the Borel summation [20].

The critical index η has the following $\epsilon-$expansion, up to the third order:

$$\eta = \frac{n+2}{2(n+8)^2}\epsilon^2 + \frac{n+2}{8(n+8)^4}(272 + 56n - n^2)\epsilon^3, \tag{82}$$

and the corresponding $n+2-$expansion, up to the third order terms, can be obtained :

$$\eta = a_1 z + a_2 z^2 + a_3 z^3; \qquad a_1 = \frac{25}{864}, \quad a_2 = -\frac{23}{2592}, \quad a_3 = \frac{43}{31104}.$$

Thus, the following approximations result

$$\begin{aligned} \eta_1(z) &= a_1 z, \\ \eta_2(z) &= a_1 z + a_2 z^2, \\ \eta_3(z) &= a_1 z + a_2 z^2 + a_3 z^3. \end{aligned}$$

The multipliers $m_2(z,s)$ and $m_3(z,s)$ at $z=2$ satisfy the condition $\mid m_2(z,s) \mid < 1, \mid m_3(z,s) \mid < 1$ for arbitrary s, the former satisfying the condition $\mid m_2(z,s) \mid = 0$ at $s = 0.586$, the latter becoming minimal at $s = 0$. Thus, we find

$$\eta_2^*(z) = a_1 z \frac{1}{(1 - \frac{a_2}{a_1(1+s)}z)^{1+s}}, \qquad \eta_3^*(z) = \frac{\eta_2(z)}{(1 - 2\frac{a_3}{a_1^3}\eta_2(z)^2)^{\frac{1}{2}}},$$

with $\eta_2^*(z=2)=0.034$, $\eta_3^*(z=2)=0.023$. The former value is very close to the so-called unconstrained $\epsilon-$expansion ($\eta=0.031\pm 3$ [26]) and the constrained $\epsilon-$expansion ($\eta=0.0320\pm 25$ [26]). The latter value approaches closely the result of Borel summation, $\eta=0.027\pm 4$ [26]. The scaling law $\frac{1}{2}\nu(1+\eta)=\beta$ is ideally satisfied with our $\nu_3^*=0.588$, $\eta_3^*=0.023$ and $\beta_3^*=0.301$.

It is also worth noting that a non-optimal, but stable trajectory for $\eta_3^*(z, s\to\infty)=\eta_2(z)\exp(\frac{a_3}{a_1}z)$ leads us to the point $\eta_3^*(2,\infty)=0.027$, which is exactly the value of the Borel summation.

The $\epsilon-$expansion for the index γ, up to the second order, has the form

$$\gamma=1+\frac{n+2}{2(n+8)}\epsilon+\frac{n+2}{4(n+8)^3}(n^2+22n+52)\epsilon^2, \tag{83}$$

and the $n+2-$expansion, up to the third order, is

$$\gamma=1+a_1z+a_3z^3; \qquad a_1=\frac{7}{72}, \quad a_3=-\frac{1}{216}.$$

The multiplier at $z=2$ acquires its minimum value at $s=0$ and

$$\gamma_3^*(z)=\frac{a_1z}{(1-\frac{2a_3}{a_1}z^2)^{1/2}}+1$$

with $\gamma_3^*(z=2)=1.165$, and with the percentage error $\epsilon_3^*(\gamma)=0.345\%$ as compared to the results of Borel summation [26]. Moving along a non-optimal but stable trajectory with $s\to\infty$, we come to $\gamma_3^*(z, s\to\infty)=a_1z\exp(\frac{a_3}{a_1}z^2)+1$, which yields the value $\gamma_3^*(2, s\to\infty)=1.161$, in the complete agreement with the Borel summation [20,26]. The scaling law $\nu(2-\eta)=\gamma$ gives for $\nu=\nu_3^*$, $\eta=\eta_3^*$ the value $\gamma=1.162$. The percentage error in this case equals 0.258%.

For the critical index α, the $\epsilon-$expansion, up to the second order, reads:

$$\alpha=\frac{4-n}{2(n+8)}\epsilon-\frac{(n+2)^2}{4(n+8)^3}(n+28)\epsilon^2, \tag{84}$$

and the $n+2-$expansion, up to the third order, follows:

$$\alpha=\frac{1}{2}-a_1z-a_2z^2-a_3z^3, \qquad a_1=\frac{1}{6}, \quad a_2=\frac{1}{432}, \quad a_3=-\frac{1}{108}.$$

For the quantity $\widetilde{\alpha} = \frac{1}{2} - \alpha$, with the set of approximations

$$\widetilde{\alpha_1}(z) = a_1 z,$$
$$\widetilde{\alpha_2}(z) = \widetilde{\alpha_1}(z) + a_2 z^2,$$
$$\widetilde{\alpha_3}(z) = \widetilde{\alpha_2}(z) + a_3 z^3,$$

following the conventional prescriptions, we find the solution corresponding to the most stable trajectory with $s = 0$:

$$\widetilde{\alpha_3}^*(z) = \frac{\widetilde{\alpha_2}(z)}{[1 - 2(a_3/a_1^3)\widetilde{\alpha_2}(z)]^{1/2}},$$

and $\alpha_3^*(z = 2) = 0.217$. The field-theory Borel summation results are not available. From the scaling law $\nu = \frac{(2-\alpha)}{3}$, we find that the value $\nu = 0.594$, corresponding to α_3^*, within the percentage error of 1% agrees with the Borel summation and our own estimate for ν. Therefore, our estimate for α may be considered as satisfactory.

The critical index δ, with the $n = -2$ limit violated by the Wilson $\epsilon-$expansion was estimated in Section 6. We found that $\delta(n = 0) = 4.882$. Theoretical field Borel summation data are not available for comparison. From our estimate $\eta_3^* = 0.023$ and the scaling relation $\delta = \frac{5-\eta}{1+\eta}$, we obtain $\delta = 4.865$, and within the percentage error of 0.349% both our estimates agree.

We believe that both good and reliable estimates can be obtained only from expansions possessing correct limits by design. Examples of such behavior are given by the $\epsilon-$expansion and field theory expansion in powers of an interaction constant.

IX. APPLICATION TO $\epsilon-$EXPANSION

In accordance with all our previous remarks, concerning the importance of the correct limits at $n = -2$ and $n \to \infty$, we considered the well-known Wilson $\epsilon-$expansion [1] and found that it does not obey these limits for the critical indices δ, γ, α and ν.

We have seen in Section 6, that for the critical index δ both limits are violated with a percentage error equal to -10%. The critical index γ (see (83)) does obey the $n = -2$

limit, but at $n \to \infty,\ \gamma = 1.75$, and the percentage error equals -12.5%, when compared with the exact $\gamma = 2$. The critical index α (see (84), obeys the $n = -2$ limit, but at $n \to \infty,\ \alpha = -0.75$, instead of the exact $\alpha = -1$, with the percentage error -25%. The critical index ν (see (76)) obeys the $n = -2$ limit, but at $n \to \infty,\ \nu = 0.875$, with the percentage error -14.286% when compared to the exact $\nu = 1$. Clearly these discrepancies should lead to an uncontrolled error within the physical region $n = 0, 1, 2, 3$.

Fortunately, the last two indices, η and β (see (82) and (79)) do obey the $n = -2$ and $n \to \infty$ limits: $\eta = 0$ at $n = -2$, $n \to \infty$ and $\beta = \frac{1}{4}$ at $n = -2$, and $\beta = \frac{1}{2}$ at $n \to \infty$.

Compare, in the physical region, the values of δ, γ, α and ν obtained from η and β by means of the scaling laws with those obtained by the direct use of the Wilson $\epsilon-$expansion. The percentage deviation from the initial Wilson expansion, for the index δ is (in this Section, in order not to cause confusion, we use the word "error" instead of the letter "ϵ" in the formula (59)):

$$E(\delta, n = 0) = 7.7\%, \qquad E(\delta, n = 1) = 7.21\%,$$
$$E(\delta, n = 2) = 7.06\%, \qquad E(\delta, n = 3) = 7.082\% \quad .$$

For the index γ :

$$E(\gamma, n = 0) = 0.7\%, \qquad E(\gamma, n = 1) = 1.97\%,$$
$$E(\gamma, n = 2) = 3.27\%, \qquad E(\gamma, n = 3) = 4.49\%,$$

the error increases considerably with n, while at $n = 0$ the correct limit $n = -2$ still favorably influences the results. For the index α we have

$$E(\alpha, n = 0) = -0.214\%, \qquad E(\alpha, n = 1) = -20.36\%,$$
$$E(\alpha, n = 2) = 168.95\%, \qquad E(\alpha, n = 3) = 52.19\%,$$

again being negligible for $n = 0$ but growing with n. For the index ν we get

$$E(\nu, n = 0) = 1.67\%, \qquad E(\nu, n = 1) = 3.13\%,$$
$$E(\nu, n = 2) = 4.52\%, \qquad E(\nu, n = 3) = 5.75\%.$$

We see that only at $n = 0$ the results possess a reasonable accuracy, and the quality of the Wilson $\epsilon-$expansion is good enough to reach, e.g. by means of $n + 2-$expansion (see Section 8), the quality of the best known estimates.

We conclude that all attempts to improve the critical indices $\delta, \gamma, \alpha, \nu$ for $n = 1, 2, 3$ directly from the Wilson $\epsilon-$expansion will leave us with an uncontrollable error. It seems reasonable to renormalize self-similarly only the critical indices η and β possessing correct limiting values and to calculate all other indices form the scaling laws.

For the index η (see (82)), using ϵ as a parameter for renormalization, the following approximations are available

$$\eta_2(\epsilon) = a_2(n)\epsilon^2, \qquad a_2(n) = \tfrac{n+2}{2(n+8)^2},$$
$$\eta_3(\epsilon) = \eta_2(\epsilon) + a_3(n)\epsilon^3, \quad a_3(n) = \tfrac{n+2}{8(n+8)^4}(272 + 56n - n^2).$$

The multiplier $m_3(n, s)$ at $\epsilon = 1$ is always minimal at $s \to \infty$, therefore

$$\eta_3^*(\epsilon) = a_2(n)\epsilon^2 \exp\{\frac{a_3(n)}{a_2(n)}\epsilon\ t^*\}. \tag{85}$$

For $t^* = 1$ we obtain

$$\eta_3^*(n = 0) = 0.045, \quad \eta_3^*(n = 1) = 0.051,$$
$$\eta_3^*(n = 2) = 0.052, \quad \eta_3^*(n = 3) = 0.05.$$

These values are probably too large. The situation may be improved if (85) is optimized using the knowledge of the exact $\eta = 0.2083$ for $\epsilon = 2,\ n = 0$ (2-d random walks) [27].

Setting $\eta_3^*(\epsilon = 2, n = 0, t^*) = 0.2083$ we find that $t^* = 0.567$ and

$$\eta_3^*(n = 0) = 0.029, \quad \eta_3^*(n = 1) = 0.033,$$
$$\eta_3^*(n = 2) = 0.034, \quad \eta_3^*(n = 3) = 0.034,$$

in a perfect agreement with the best estimates by the Borel summation [20]. If t^* is optimized from the knowledge of the $2d$ Ising model exact $\eta = 0.25$, then similarly, $t^* = 0.603$ and

$$\eta_3^*(n = 0) = 0.03, \quad \eta_3^*(n = 1) = 0.034,$$
$$\eta_3^*(n = 2) = 0.035, \quad \eta_3^*(n = 3) = 0.035.$$

There is also another way to get information from the $\epsilon-$expansion for the index η. Proceed similarly to the case of $\frac{1}{n}-$expansion, and re-expand (82) in powers of $y = \frac{n+2}{(n+8)^2}$ (at $\epsilon = 1$). Then, up to the second order in y,

$$\eta \approx \frac{3}{8}y + 9y^2$$

and $\eta_2^* = \frac{3}{8} y \exp(24y)$, with

$$\eta_2^*(n=0) = 0.025, \quad \eta_2^*(n=1) = 0.034,$$
$$\eta_2^*(n=2) = 0.039, \quad \eta_2^*(n=3) = 0.042,$$

still in a good agreement with a set of data available for the critical index η [21].

For the critical index β, Eq.(79) defines the following set of approximations:

$$\beta_0(\epsilon) = a_0, \qquad a_0 = 1,$$
$$\beta_1(\epsilon) = \beta_0(\epsilon) + a_1(n)\epsilon, \qquad a_1(n) = -\frac{3}{2(n+8)},$$
$$\beta_2(\epsilon) = \beta_1(\epsilon) + a_2(n)\epsilon^2, \qquad a_2(n) = \frac{(n+2)(2n+1)}{2(n+8)^3}.$$

The multiplier

$$m_2(n,s) = 1 + \frac{a_1(n)}{a_0}\frac{1+s}{s}\epsilon + \frac{a_2(n)}{a_0}\frac{2+s}{s}\epsilon^2$$

is equal to zero at $s = -\frac{2a_2+a_1}{a_2+a_1+a_0}$, and the evolution integral gives

$$\beta_2^*(\epsilon) = \frac{\beta_1(\epsilon)}{\epsilon^s}\frac{1}{[1 - \frac{2a_2^2}{s}a_0^{-(1+\frac{s}{2})}\beta_1^{\frac{2}{s}}(\epsilon)]^{\frac{s}{2}}}. \tag{86}$$

From here, at $\epsilon = 1$,

$$\beta_2^*(n=0) = 0.313, \quad \beta_2^*(n=1) = 0.333,$$
$$\beta_2^*(n=2) = 0.35, \quad \beta_2^*(n=3) = 0.364.$$

The formula (86) is applicable only up to $n = 42$, where $s \to 0$ and (86) becomes undefined. For $n \geq 42$, the exponential summation is optimal:

$$\beta_2^*(\epsilon) = \beta_1(\epsilon)\exp(\frac{a_2(n)}{a_0}\epsilon), \quad n \geq 42.$$

For $n = 2, 3$, β_2^* almost coincides with the Borel summation values 0.346 ± 2 and 0.365 ± 2, respectively, being larger for $n = 0, 1$, where the Borel summation gives 0.302 ± 15, and 0.325 ± 1 respectively.

The $\epsilon-$expansion was obtained also from the theoretical field approach up to the fifth order in ϵ [28]. For the critical index ν, up to the second order in ϵ, one has

$$V \equiv \nu^{-1} = a_a(n) + a_1(n)\epsilon + a_2(n)\epsilon^2,$$
$$a_a(n) = 2, \quad a_1(n) = -\frac{n+2}{n+8}, \quad a_2(n) = -\frac{(n+2)(13n+44)}{2(n+8)^3}, \tag{87}$$

the limits $n = -2$ and $n \to \infty$ being satisfied, in distinction from the original $\epsilon-$expansion. We use the following approximations to the quantity $\widetilde{V} = -V + a_0(n)$:

$$\begin{aligned} \widetilde{V}_1(\epsilon) &= -a_1(n)\epsilon, \\ \widetilde{V}_2(\epsilon) &= -a_1(n)\epsilon - a_2(n)\epsilon^2. \end{aligned} \tag{88}$$

All terms in (88) are positive and, the optimal renormalization corresponds to $s \to \infty$. The evolution integral can be readily calculated giving

$$\widetilde{V}_2^*(\epsilon) = -a_1(n)\epsilon \exp\{\frac{a_2(n)}{a_1(n)}\epsilon\ t^*\}. \tag{89}$$

We found that at $t^* = 1,\ \epsilon = 1,$

$$\begin{aligned} \nu_2^*(n=0) = 0.607, \quad \nu_2^*(n=1) = 0.655, \\ \nu_2^*(n=2) = 0.698, \quad \nu_2^*(n=3) = 0.736. \end{aligned}$$

Similarly to the case of the index η, the self-similarly renormalized $\epsilon-$expansion for ν overestimates the critical index, as may be seen from the comparison with other results [21]. Let us optimize the expression for ν_2^* using the knowledge of the $2d$ Ising $\nu = 1$ [29]. From the condition $\nu_2^*(\epsilon = 2, n = 1, t^*) = 1$ we find $t^* = 0.576$ and the optimized values

$$\begin{aligned} \nu_2^*(n=0) = 0.59, \quad \nu_2^*(n=1) = 0.628, \\ \nu_2^*(n=2) = 0.662, \quad \nu_2^*(n=3) = 0.691. \end{aligned}$$

Compared to the best known calculations of ν from the Borel summation and similar methods, for $n = 0, 1$ our estimates practically coincide with them, and for $n = 2, 3$ the percentage errors are -0.451% compared to $\nu = 0.665$ at $n = 2$ [30] and 1.003% compared to $\nu = 0.698$ at $n = 3$ [30].

It is interesting that by a single parameter optimization the index ν is improved in the whole physical range. It is worth noting that t^* used for optimization is about the same for η and ν. Evaluation of the critical indices is also possible basing on the field-theoretical expansions. Information is obtained in the latter case from the perturbative series directly and the results are marginally sensitive to the way in which the position of an infrared stable fixed point is determined [20,31].

X. APPLICATION TO THE FIELD THEORY EXPANSION

Theoretical field approach in the theory of critical phenomena gives, probably, the most accurate and consistent estimates for the critical indices η and γ [2]. The analysis of the expansions in powers of the interaction constant g (g-expansion) for these indices from the viewpoint of the limiting cases $n = -2,\ n \to \infty$, became possible only when the g-expansions have been written for arbitrary n [30]. By direct inspection of the expressions for η and γ from [30], we found that $n \to \infty$ limit is obeyed rigorously if $g = 1$, i.e $\eta = 0,\ \gamma = 2$ and the $n = -2$ limit is obeyed with very high accuracy for arbitrary g, i.e $\eta \approx 0\ \gamma \approx 1$ with the error insignificant within the framework of the theory of critical indices.

The standard approach [20] uses, for computing the infrared stable fixed point g^* of the beta-function $W(g)$, a complicated Borel summation technique. Then critical indices are calculated as $\gamma(g^*),\ \eta(g^*)$. This approach requires a number of terms in the expansions. We apply below the self-similar renormalization to only the initial three terms in the expressions for $W(g), \gamma^{-1}(g),\ \eta(g)$ and obtain estimates with an accuracy comparable to the best known Borel summation results obtained from all known terms in the expansions.

We construct the following set of approximations to $W(g)$ using the $g-$expansion from [30]:

$$\begin{aligned} W_2(g) &= -g + g^2, \\ W_3(g) = W_2(g) - a_3(n)\ g^3, &\qquad a_3(n) = \tfrac{6.07407408\ n + 28.14814815}{(n+8)^2}. \end{aligned} \tag{90}$$

From the equation $W_2(g) = f$ we find the expansion function $x(f) = \frac{1}{2}(1 + \sqrt{1 + 4f}\ )$. The points of the approximation cascade trajectory are

$$\begin{aligned} y_2(f) &= f, \\ y_3(f) &= y_2(f) - \tfrac{a_3}{8} x^3(f). \end{aligned}$$

For the velocity field we get

$$v_3(f) = -\frac{a_3}{8} x^3(f).$$

By analogy with Section 4, the evolution-integral solution for W_3^* is obtained implicitly from the equation

$$\int_{W_2(g)}^{W_3^*} \frac{df}{v_3(f)} = t^*,$$

and the root g^* of the equation $W_3^*(g, n, t^* = 1) = 0$ is obtained numerically, as a function $g^* = g^*(n)$. In the physically important cases

$$g^*(n = 0) = 1.59, \quad g^*(n = 1) = 1.559,$$
$$g^*(n = 2) = 1.524, \quad g^*(n = 3) = 1.491.$$

At $n = -2$, $g^* = 1.599$ and at $n \to \infty$, $g^* = 1$. The dependence of $g^*(n)$ in the interval $n = -2, 0$ is nonmonotonous, a maximum is reached at $n = -1$, where $g^* = 1.61$. Our values are higher than the results of [30], but remain within the reasonable bounds and show the same tendency, at least for $n = 0, \infty$. No data are available for comparison for $n = -2, 0$.

For the critical index η, keeping the starting two terms in powers of g, we can write down the following approximations

$$\begin{aligned} \eta_2(g) = b_2(n)g^2, \qquad b_2(n) = \tfrac{0.2962962963(n+2)}{(n+8)^2}, \\ \eta_3(g) = \eta_2(g) + b_3(n)g^3, \quad b_3(n) = \tfrac{0.0246840014n^2+0.246840014n+0.3949440224}{(n+8)^3}, \end{aligned} \tag{91}$$

and the evolution integral can be readily calculated resulting in

$$\eta_3^*(g, n) = b_2(n)g^2 \exp\{\frac{b_3(n)}{b_2(n)}g\},$$

and

$$\eta_3^*(n = 0) = 0.027, \quad \eta_3^*(n = 1) = 0.030,$$
$$\eta_3^*(n = 2) = 0.031, \quad \eta_3^*(n = 3) = 0.031.$$

These values are practically the same that quoted in [20] where the Borel summation has been used: $\eta(n = 0) = 0.026 \pm 3$, $\eta(n = 1) = 0.031 \pm 4$, $\eta(n = 2) = 0.033 \pm 4$, $\eta(n = 3) = 0.033 \pm 4$. At $n = 4$, η_3^* decreases to 0.03, the same tendency is seen in the data of [30].

For the critical index γ we keep the three starting terms in powers of g :

$$G \equiv \gamma^{-1} = 1 + c_1(n)g - c_2(n)g^2, \qquad c_1(n) = -\frac{n+2}{2(n+8)}, \quad c_2(n) = -\frac{n+2}{(n+8)^2}. \tag{92}$$

The following approximations to the quantity $\widetilde{G} = -G + 1$ are used:

$$\widetilde{G}_1(g) = c_1(n)g,$$
$$\widetilde{G}_2(g) = \widetilde{G}_1(g) - c_2(n)g^2.$$

The multiplier $m_2(g, n, s) = 1 - \frac{c_2(n)}{c_1(n)}\frac{2+s}{1+s}g$, calculated for $g = g^*(n)$, is always minimal for $s = 0$. The evolution integral gives

$$\widetilde{G}_2^*(g, n) = c_1(n)g\frac{1}{1 + \frac{c_2(n)}{c_1(n)}g}.$$

For γ_2^* we obtain

$$\gamma_2^*(n = 0) = 1.166, \quad \gamma_2^*(n = 1) = 1.239,$$
$$\gamma_2^*(n = 2) = 1.305, \quad \gamma_2^*(n = 3) = 1.363.$$

The percentage errors are $\epsilon_2(\gamma(n = 0)) = 0.431\%$, as compared to 1.161 from the Borel summation: zero error as compared to 1.241 ± 2 at $n = 1$; $\epsilon_2(\gamma(n = 2)) = -0.836\%$ as compared to 1.316, and $\epsilon_2(\gamma(n = 3)) = -1.659\%$, as compared to 1.386 from the Borel summation. In the latter case of $n = 3$ we also constructed the set of approximations directly for the index γ, expanding (92) in powers of g, up to the third order :

$$\gamma = 1 + d_1(n)g + d_2(n)g^2, \qquad d_1(n) = -c_1(n), \quad d_2(n) = c_1^2(n) + c_2(n). \tag{93}$$

Approximating $\widetilde{\gamma} = \gamma - 1$ by

$$\widetilde{\gamma_1}(n) = d_1(n)g,$$
$$\widetilde{\gamma_2}(n) = \widetilde{\gamma_1}(n) + d_2(n)g^2,$$

and finding out that the multiplier $m_2(g, n, s)$ is minimal at $s \to \infty$, we have

$$\widetilde{\gamma_2}^*(n) = d_1(n)g\exp\{\frac{d_2(n)}{d_1(n)}g\ t^*\}. \tag{94}$$

At $t^* = 1$ we obtain $\gamma_2^*(n = 3) = 1.37$ and the percentage error -1.154%. The same procedure applied to $n = 0, 1, 2$ always keeps the error $\epsilon < 1\%$. These results are quite accurate, especially if to remember that only the starting terms were used. Unfortunately,

the expression (93) does not obey the $n \to \infty$ limit with a percentage error -3.7%. An optimization of the expression (94) may be carried out requiring that the limit $\gamma = 2$ at $n \to \infty$ be restored (see also section 5). This aim is achieved with $t^* = 1.15$ and $\gamma_2^*(n = 3) = 1.375$ with a percentage error -0.794%. An effective increase of t^* mimics the effect of introducing higher terms into consideration.

XI. CONCLUSION

We suggested here a new variant in the method of self-similar renormalization permitting to find effective sums of asymptotic series. The advantage of the method is that it allows to get results exploiting just a few first terms of given series, when no other resummation techniques work. At the same time, the accuracy of the results is not worse than that reached in other known sophisticated techniques involving about ten terms. In addition, our method in the majority of cases, makes it possible to present answers in the form of simple analytical expressions that are easy to study for considering their dependencies on various parameters, including asymptotic behavior with respect to these parameters.

The possibility of realizing a renormalization, having in hands only a few terms of a series, is due to an algebraic transformation which is equivalent to the effective increase of approximation orders. That is why we call this variant the algebraic self-similar renormalization.

The general idea of the self-similar renormalization [4-8] is to extract the maximum of information from the minimal number of terms. Such a minimax criterion, certainly, can be followed only with the help of additional functions making the convergence as fast as possible. These functions are called control or governing functions, and they play the same role as the control functions in the optimal control theory. In the algebraic self-similar renormalization, the control functions are introduced into powers of an algebraic transformation. The choice of these control functions is based on the principle of maximal stability by minimizing the absolute value of mapping multipliers.

We illustrated the effectiveness of our approach by renormalizing divergent series in the theory of critical phenomena. Doing this, we specially restricted ourselves from using many terms of perturbative series, which are sometimes known–This is to emphasize that our approach is effective when, really, only a minimal information is available. Dealing with higher-order terms needs a multiple repetition of the renormalization procedure. This requires a slight generalization of the technique and much more place for presentation. This multiple renormalization will be the subject of a separate publication.

The method suggested is quite general and can be applied to arbitrary divergent series. The choice of examples from the theory of critical phenomena owes to the common interest to this problem. We wanted also to stress that even for the problem, where so much is known, there are ways of improvement by first, obtaining the results much easier, second, deriving analytical formulas, not involving heavy numerical calculations, and, finally, by restoring correct asymptotic behavior with respect to physical parameters, such as the number of components.

Another important message which we would like to bring up to readers is that one should not be afraid of simple perturbative series that, being divergent, seem, at the first glance, to be senseless: Even a seemingly bad perturbative series contains quite a lot of useful information which can be extracted by means of an efficient renormalization procedure. We hope we were able to convince the reader that the algebraic self-similar renormalization suggested can be such a tool for extracting quite accurate information even from bad and short series.

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